

A Central Limit Theorem For Linear Random Fields

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1 Introduction

Consider a two-dimensional, linear random field, say

$$X_{j,k} = \sum_{r \in \mathbb{Z}} \sum_{s \in \mathbb{Z}} a_{r,s} \xi_{j-r,k-s} = \sum_{r \in \mathbb{Z}} \sum_{s \in \mathbb{Z}} a_{j+r,k+s} \xi_{-r,-s},$$

where $a_{r,s}$, $r, s \in \mathbb{Z}$, are square summable, $\xi_{r,s}$, $r, s \in \mathbb{Z}$, are i.i.d. with mean 0 and unit variance, and \mathbb{Z} denotes the integers. It is convenient to regard the array $\mathbf{a} = (a_{r,s} : r, s \in \mathbb{Z})$ as an element of $\ell^2(\mathbb{Z}^2)$. Let F denote the common distribution function of the $\xi_{r,s}$ and (Ω, \mathcal{A}, P) the probability space on which they are defined. If Γ is a finite subset of \mathbb{Z}^2 , let

$$S = S(\mathbf{a}, \Gamma) = \sum_{(j,k) \in \Gamma} X_{j,k}$$

and

$$\sigma^2 = \sigma^2(\mathbf{a}, \Gamma) = E(S^2),$$

and suppose that $\sigma^2 > 0$. (Of course, S depends on $\omega \in \Omega$ too, but this dependence is suppressed. As indicated dependence on \mathbf{a} and Γ will only be displayed when needed for clarity.) Then

$$S = \sum_{r \in \mathbb{Z}} \sum_{s \in \mathbb{Z}} b_{r,s} \xi_{-r,-s},$$

where

$$b_{r,s} = b_{r,s}(\mathbf{a}, \Gamma) = \sum_{(j,k) \in \Gamma} a_{j+r,k+s},$$

and

$$\sigma^2 = \sum_{r \in \mathbb{Z}} \sum_{s \in \mathbb{Z}} b_{r,s}^2,$$

assumed to be positive. Let Φ denote the standard normal distribution and

$$G(z) = G(z; \mathbf{a}, \Gamma, F) = P \left[\frac{S}{\sigma} \leq z \right], \quad z \in \mathbb{R}.$$

Sufficient conditions for G to be close to Φ are developed.

There has been some recent work on the Central Limit Theorem for sums of linear processes with values in a Hilbert space [5]. Other recent work on the Central Limit Theorem for linear random fields has emphasized the Beveridge Nelson decomposition, developed by Phillips and Solo [7] for processes. This approach leads to functional versions of the CLT, [4], [6], and their references. Our approach follows that of Ibragimov [3]. For the case in which Γ is a rectangle, it requires no additional conditions on the coefficients, but does require the innovations to be independent and does not deliver a functional version.

2 Generalities

Let

$$\rho = \rho(\mathbf{a}, \Gamma) = \max_{r,s \in \mathbb{Z}} \frac{|b_{r,s}|}{\sigma}. \quad (1)$$

Interest in ρ stems from the following:

Proposition 1 *Let \mathcal{H} denote a class of distribution functions for which*

$$\int_{\mathbb{R}} xH\{dx\} = 0, \quad \int_{\mathbb{R}} x^2H\{dx\} = 1, \text{ for all } H \in \mathcal{H}, \quad (2)$$

$$\lim_{c \rightarrow \infty} \sup_{H \in \mathcal{H}} \int_{|x| > c} x^2H\{dx\} = 0; \quad (2)$$

Then $\forall \epsilon > 0, \exists \delta = \delta_{\epsilon, \mathcal{H}}$, depending only on ϵ and \mathcal{H} for which for which

$$d(G, \Phi) := \sup_z |G(z) - \Phi(z)| \leq \epsilon \quad (3)$$

for all all $F \in \mathcal{H}$ for all arrays \mathbf{a} and regions $\Gamma \subset \mathbb{Z}^2$ for which $\rho \leq \delta$.

Proof. Let $\hat{\cdot}$ denote Fourier transform (characteristic function), so that $\hat{F}(t) = \int_{\mathbb{R}} e^{itx} F\{dx\}$, and

$$L(\eta) = \frac{1}{\sigma^2} \sum_{r,s \in \mathbb{Z}} \int_{|b_{r,s}x| > \eta\sigma} |b_{r,s}|^2 F\{dx\}$$

for $\eta > 0$. Then, for any $\eta > 0$, $|\hat{G}(t) - \hat{\Phi}(t)| \leq \eta|t|^3/6 + t^2L(\eta)$ for all $t \in \mathbb{R}$ from the proof of the Central Limit Theorem for independent summand ([1], pp. 359 - 361), and

$$\begin{aligned} \sup_z |G(z) - \Phi(z)| &\leq \frac{1}{\pi} \int_{-T}^T \left| \frac{\hat{G}(t) - \hat{\Phi}(t)}{t} \right| dt + \frac{24}{\pi\sqrt{2\pi}T} \\ &\leq \frac{\eta T^3}{18} + \frac{1}{2}T^2L(\eta) + \frac{24}{T} \end{aligned} \quad (4)$$

for any $T > 0$ by the smoothing inequality ([2], pp. 510 - 512). Given ϵ , let $T_\epsilon = \frac{96}{\epsilon}$ and $\eta_\epsilon = 4\epsilon/T_\epsilon^3$. Then the left side of (4) is at most $\frac{1}{2}T_\epsilon^2L(\eta_\epsilon) + \frac{1}{2}\epsilon$. Next, let $J(c) = \sup_{H \in \mathcal{H}} \int_{|x|>c} x^2 H\{dx\}$ for $c > 0$, so that $J(c) \rightarrow 0$ as $c \rightarrow \infty$ by (2); and let $J^\#(z) = \inf\{c > 0 : J(c) \leq z\}$ for $z > 0$. Then

$$L(\eta) \leq \sum_{r \in \mathbb{Z}} \sum_{s \in \mathbb{Z}} b_{r,s}^2 \int_{|x|>\eta/\rho} x^2 F\{dx\} \leq J\left(\frac{\eta}{\rho}\right),$$

and $\delta = \eta_\epsilon/J^\#(T_\epsilon^{-2}\epsilon)$ has the desired properties. \diamond

Next, let

$$\|\mathbf{a}\|_p = \left[\sum_{r,s \in \mathbb{Z}} |a_{r,s}|^p \right]^{\frac{1}{p}} \leq \infty$$

for $1 \leq p \leq 2$. Thus, $\|\mathbf{a}\|_2$ is assumed to be finite and $\|\mathbf{a}\|_p$ may be finite for some value of $p < 2$. In terms of $\|\mathbf{a}\|_p$ there is a simple bound on ρ ,

$$\rho \leq \frac{\|\mathbf{a}\|_p \Gamma^{\frac{1}{q}}}{\sigma}, \quad (5)$$

where q denotes the conjugate, $1/p + 1/q = 1$ and $\#\Gamma$ denotes the cardinality of Γ . In particular, $\rho \leq \|\mathbf{a}\|_1/\sigma$. This leads to:

Corollary 1 *Let \mathcal{H} be as in Proposition 1. If $\|\mathbf{a}\|_1 < \infty$, then $\forall \epsilon > 0$, $\exists \kappa = \kappa(\epsilon, \mathcal{H}) > 0$ for which (3) holds whenever $\sigma \geq \kappa\|\mathbf{a}\|_1$ and $F \in \mathcal{H}$.*

Proof. For δ as per in Proposition 1, pick κ such that $\kappa\delta \leq 1$. The result is then an easy consequence of the proposition and the fact that $\rho \leq \|\mathbf{a}\|_1/\sigma$. \diamond

3 Rectangles

To bound ρ , suppose that the maximum occurs when $r = r_0$ and $s = s_0$, say

$$|b_{r_0,s_0}| = \max_{r,s} |b_{r,s}|,$$

and let $\Delta b_{u,v} = b_{u,v} - b_{u,v-1} - b_{u-1,v} + b_{u-1,v-1}$ for $(u, v) \in \mathbb{Z}^2$. Then

$$b_{r_0+r, s_0+s} - b_{r_0, s_0+s} - b_{r_0+r, s_0} + b_{r_0, s_0} = \sum_{r=r_0+1}^{r_0+r} \sum_{v=s_0+1}^{s_0+s} \Delta b_{u,v} \quad (6)$$

for $r, s \geq 1$. Let

$$Q_{m,n} = \sum_{r=1}^m \sum_{s=1}^n \sum_{u=r_0+1}^{r_0+r} \sum_{v=s_0+1}^{s_0+s} |\Delta b_{u,v}| = \sum_{r=r_0+1}^{r_0+m} \sum_{s=s_0+1}^{s_0+n} (r - r_0)(s - s_0) |\Delta b_{r,s}| \quad (7)$$

for $m, n \geq 1$. Then $|b_{r_0, s_0}| \leq |b_{r_0, s_0+s}| + |b_{r_0+r, s_0}| + |b_{r_0, s_0}| = \sum_{r=r_0+1}^{r_0+r} \sum_{v=s_0+1}^{s_0+s} |\Delta b_{u,v}|$ for all $r, s \geq 1$ and, therefore,

$$mn|b_{r_0, s_0}| \leq \sum_{r=1}^m \sum_{s=1}^n (|b_{r_0+r, s_0+s}| + |b_{r_0, s_0+s}| + |b_{r_0+r, s_0}|) + Q_{m,n}$$

for all $m, n \geq 1$. Here

$$\sum_{r=1}^m \sum_{s=1}^n |b_{r_0+r, s_0+s}| \leq \sqrt{mn} \sqrt{\sum_{r=1}^m \sum_{s=1}^n b_{r_0+r, s_0+s}^2} \leq \sqrt{mn} \sigma,$$

and similarly, $\sum_{r=1}^m \sum_{s=1}^n |b_{r_0, s_0+s}| \leq m\sqrt{n}\sigma$ and $\sum_{r=1}^m \sum_{s=1}^n |b_{r_0+r, s_0}| \leq \sqrt{m}n\sigma$. So, $mn|b_{r_0, s_0}| \leq \sqrt{mn}\sigma + m\sqrt{n}\sigma + \sqrt{m}n\sigma + Q_{m,n}$. That is,

$$\rho = \frac{|b_{r_0, s_0}|}{\sigma} \leq \left(\frac{2}{\sqrt{m}} + \frac{2}{\sqrt{n}} \right) + \frac{Q_{m,n}}{mn\sigma} \quad (8)$$

for any $m, n \geq 1$. The first two terms may be made small by taking m and n large. Thus, the issue is $Q_{m,n}$. Suppose now that Γ can be written as the union of ℓ non-empty pairwise mutually exclusive rectangles,

$$\Gamma = \bigcup_{i=1}^{\ell} \{(j, k) : \underline{M}_i \leq j \leq \overline{M}_i, \underline{N}_i \leq k \leq \overline{N}_i\}. \quad (9)$$

Proposition 2 *If Γ is of the form (9), then where $M, N \geq 1$, then*

$$\rho \leq 12 \left(\frac{\sqrt{\ell} \|\mathbf{a}\|_2}{\sigma} \right)^{\frac{1}{5}} + \frac{8\sqrt{\ell} \|\mathbf{a}\|_2}{\sigma}. \quad (10)$$

Proof. In this case $b_{r,s} = \sum_{i=1}^{\ell} b_{r,s}^{(i)}$, where $b_{r,s}^{(i)} = \sum_{j=\underline{M}_i}^{\overline{M}_i} \sum_{k=\underline{N}_i}^{\overline{N}_i} a_{j+r, k+s}$, $\Delta b_{r,s} = \sum_{i=1}^{\ell} \Delta b_{r,s}^{(i)}$, and $\Delta b_{r,s}^{(i)} = a_{r+\overline{M}_i, s+\overline{N}_i} - a_{r+\underline{M}_i, s+\overline{N}_i} - a_{r+\overline{M}_i, s+\underline{N}_i} + a_{r+\underline{M}_i, s+\underline{N}_i}$. So,

$$\begin{aligned} Q_{m,n} &\leq mn \sum_{r=1}^m \sum_{s=1}^n \sum_{i=1}^{\ell} (|a_{r+\overline{M}_i, s+\overline{N}_i} - a_{r+\underline{M}_i, s+\overline{N}_i} - a_{r+\overline{M}_i, s+\underline{N}_i} + a_{r+\underline{M}_i, s+\underline{N}_i}|) \\ &\leq 4(mn)^2 \sqrt{\ell} \|\mathbf{a}\|_2 \end{aligned}$$

by Schwartz' Inequality, and

$$\rho \leq \left(\frac{2}{\sqrt{m}} + \frac{2}{\sqrt{n}} \right) + \frac{4mn\sqrt{\ell}\|\mathbf{a}\|_2}{\sigma}$$

for any $m, n \geq 1$. Letting $m = n = \lceil (\sigma/\sqrt{\ell}\|\mathbf{a}\|_2)^{\frac{4}{5}} \rceil$, the least integer that exceeds $(\sigma/\sqrt{\ell}\|\mathbf{a}\|_2)^{\frac{4}{5}}$ then leads to (10). \diamond

When specialize to (intersections of) rectangles (with \mathbb{Z}^2), the proposition provides a complete analogue of Ibragimov's theorem [3] with lots of uniformity.

Corollary 2 . *Let \mathcal{H} be as in Proposition 1 and let \mathcal{R}_κ be the collections of pairs (\mathbf{a}, Γ) for which $\|\mathbf{a}\|_2 > 0$, Γ is the a rectangle, and $\sigma(\mathbf{a}, \Gamma) \geq \kappa\|\mathbf{a}\|_2$. Then, as $\kappa \rightarrow \infty$, the distributions of S/σ converge to the Φ uniformly with respect to $(\mathbf{a}, \Gamma) \in \mathcal{R}_\kappa$ and $F \in \mathcal{H}$*

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